Probability Theory
2023/24 Period IIb
Instructor: Gilles Bonnet
Re-Exam
8/7/2024
Duration: 2 hours

First name: _____

Surname: _____

Student number: _____

This exam contains 6 problems. Enter all requested information on the top of this page.

Your answers should be written in this booklet. Avoid handing in extra paper. In case you hand in extra paper and/or write parts of your answers in non obvious places, mark this explicitly so that it is not missed out during the grading process.

Do not use red ink nor pencil.

You are **not** allowed to use any electronic devices during the exam.

You are **not** allowed to use any books, lecture notes or handwritten notes.

Each answer must be **justified**, unless explicitly stated otherwise.

Do not write on the table below.

Problem	Points	Score
1	10	
2	10	
3	15	
4	16	
5	16	
6	23	
Total:	90	

Satisfying the above instructions gives you 10 free points.

z	0	0,01	**0,02**	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0	0,5	0,50399	0,50798	0,51197	0,51595	0,51994	0,52392	0,5279	0,53188	0,53586
0,1	0,53983	0,5438	0,54776	0,55172	0,55567	0,55962	0,56356	0,56749	0,57142	0,57535
0,2	0,57926	0,58317	0,58706	0,59095	0,59483	0,59871	0,60257	0,60642	0,61026	0,61409
0,3	0,61791	0,62172	0,62552	0,6293	0,63307	0,63683	0,64058	0,64431	0,64803	0,65173
0,4	0,65542	0,6591	0,66276	0,6664	0,67003	0,67364	0,67724	0,68082	0,68439	0,68793
0,5	0,69146	0,69497	0,69847	0,70194	0,7054	0,70884	0,71226	0,71566	0,71904	0,7224
0,6	0,72575	0,72907	0,73237	0,73565	0,73891	0,74215	0,74537	0,74857	0,75175	0,7549
0,7	0,75804	0,76115	0,76424	0,7673	0,77035	0,77337	0,77637	0,77935	0,7823	0,78524
0,8	0,78814	0,79103	0,79389	0,79673	0,79955	0,80234	0,80511	0,80785	0,81057	0,81327
0,9	0,81594	0,81859	0,82121	0,82381	0,82639	0,82894	0,83147	0,83398	0,83646	0,83891
1	0,84134	0,84375	0,84614	0,84849	0,85083	0,85314	0,85543	0,85769	0,85993	0,86214
1,1	0,86433	0,8665	0,86864	0,87076	0,87286	0,87493	0,87698	0,879	0,881	0,88298
1,2	0,88493	0,88686	0,88877	0,89065	0,89251	0,89435	0,89617	0,89796	0,89973	0,90147
1,3	0,9032	0,9049	0,90658	0,90824	0,90988	0,91149	0,91309	0,91466	0,91621	0,91774
1,4	0,91924	0,92073	**0,9222**	0,92364	0,92507	0,92647	0,92785	0,92922	0,93056	0,93189
1,5	0,93319	0,93448	0,93574	0,93699	0,93822	0,93943	0,94062	0,94179	0,94295	0,94408
1,6	0,9452	0,9463	0,94738	0,94845	0,9495	0,95053	0,95154	0,95254	0,95352	0,95449
1,7	0,95543	0,95637	0,95728	0,95818	0,95907	0,95994	0,9608	0,96164	0,96246	0,96327
1,8	0,96407	0,96485	0,96562	0,96638	0,96712	0,96784	0,96856	0,96926	0,96995	0,97062
1,9	0,97128	0,97193	0,97257	0,9732	0,97381	0,97441	0,975	0,97558	0,97615	0,9767
2	0,97725	0,97778	0,97831	0,97882	0,97932	0,97982	0,9803	0,98077	0,98124	0,98169
2,1	0,98214	0,98257	0,983	0,98341	0,98382	0,98422	0,98461	0,985	0,98537	0,98574
2,2	0,9861	0,98645	0,98679	0,98713	0,98745	0,98778	0,98809	0,9884	0,9887	0,98899
2,3	0,98928	0,98956	0,98983	0,9901	0,99036	0,99061	0,99086	0,99111	0,99134	0,99158
2,4	0,9918	0,99202	0,99224	0,99245	0,99266	0,99286	0,99305	0,99324	0,99343	0,99361
2,5	0,99379	0,99396	0,99413	0,9943	0,99446	0,99461	0,99477	0,99492	0,99506	0,9952
2,6	0,99534	0,99547	0,9956	0,99573	0,99585	0,99598	0,99609	0,99621	0,99632	0,99643
2,7	0,99653	0,99664	0,99674	0,99683	0,99693	0,99702	0,99711	0,9972	0,99728	0,99736
2,8	0,99744	0,99752	0,9976	0,99767	0,99774	0,99781	0,99788	0,99795	0,99801	0,99807
2,9	0,99813	0,99819	0,99825	0,99831	0,99836	0,99841	0,99846	0,99851	0,99856	0,99861
3	0,99865	0,99869	0,99874	0,99878	0,99882	0,99886	0,99889	0,99893	0,99896	0,999
3,1	0,99903	0,99906	0,9991	0,99913	0,99916	0,99918	0,99921	0,99924	0,99926	0,99929
3,2	0,99931	0,99934	0,99936	0,99938	0,9994	0,99942	0,99944	0,99946	0,99948	0,9995
3,3	0,99952	0,99953	0,99955	0,99957	0,99958	0,9996	0,99961	0,99962	0,99964	0,99965
3,4	0,99966	0,99968	0,99969	0,9997	0,99971	0,99972	0,99973	0,99974	0,99975	0,99976
3,5	0,99977	0,99978	0,99978	0,99979	0,9998	0,99981	0,99981	0,99982	0,99983	0,99983
3,6	0,99984	0,99985	0,99985	0,99986	0,99986	0,99987	0,99987	0,99988	0,99988	0,99989
3,7	0,99989	0,9999	0,9999	0,9999	0,99991	0,99991	0,99992	0,99992	0,99992	0,99992
3,8	0,99993	0,99993	0,99993	0,99994	0,99994	0,99994	0,99994	0,99995	0,99995	0,99995
3,9	0,99995	0,99995	0,99996	0,99996	0,99996	0,99996	0,99996	0,99996	0,99997	0,99997
4	0,99997	0,99997	0,99997	0,99997	0,99997	0,99997	0,99998	0,99998	0,99998	0,99998

Table 1: CDF Φ of a standard normal distribution.
Example: the value in the row "1,4" and column "0,02" gives the approximation $\Phi(1.42) \simeq 0.9222$.

1. (10 points) How many different "words" can one form with all the letters of the word "BANANA"?
   Remark 1: Any sequence of letters is considered a word, even if it does not form a meaningful word in English.
   Remark 2: Your solution should include a short explanation of your reasoning, or the computation should be written in a way which makes it obvious to the corrector what is the reasoning, or both.
   Remark 3: In order to get full points, your final answer should be in a form of an integer number (not a formula).

2. (10 points) Let $m, n \in \mathbb{N}$. Let X be a continuous random variable with density function

$$f_X(x) = \begin{cases} nx^{n-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Let $Y = X^{1/m}$. Find the density function of Y.

3. (15 points) Let $X_1 \sim \text{Poisson}(\lambda_1)$ and $X_2 \sim \text{Poisson}(\lambda_2)$ be independent Poisson random variables. What is the distribution of $X_1 + X_2$? Justify your answer by writing down the appropriate computation.

4. A student takes an exam with $n = 192$ questions. They get the answer to each question correctly with probability $p = \frac{3}{4}$, independently of all others. In order to pass the student needs to answer correctly at least 122 questions.
   Let X_i be the indicator variable of the event that the student answers correctly the i-th question.
   Let X be the number of correct answers.
   Remark: $192 = 2^6 \times 3$.
   (a) (3 points) Compute the expected value of X.
   (b) (3 points) Compute the variance of X.
   (c) (10 points) Compute an approximation of the probability that the student passes.

5. (a) (8 points) State and prove Markov's inequality.
   (b) (8 points) State and prove Chebyshev's inequality.

6. The joint density of X and Y is given by

$$f(x,y) = C(x-y)e^{-x}\mathbf{1}(-x < y < x).$$

(a) (4 points) Find the value of the constant C.
Hint: You can use that $\int_0^\infty e^{-x}x^n \, \mathrm{d}x = n!$ for any $n \in \mathbb{N}$.

(b) (9 points) Find the density function f_Y of Y.

(c) (5 points) Find $\mathbb{E}[Y]$.

(d) (2 points) Let f_X denote the density function of X. Show that $f_X(x) > 0$ for any $x > 0$.

(e) (3 points) Determine if X and Y are independent.